Twists and Turns

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Goals and Objectives

1. Students will experience topology as the geometry of distortion.

2. Students will see topology as a deductive system of interrelated concepts.

3. Students will use patterns and inductive reasoning to summarize experimental results.

4. Students will be able to apply some topological classifications.

SUNSHINE STATE STANDARDS

The student describes, analyzes, and generalizes a wide variety of patterns, relations, and functions. (MA.D.1.2, MA.D.1.3, MA.D.1.4)

The student describes, draws, identifies, and analyzes two- and three-dimensional shapes. (MA.C.1.3, MA.C.1.4)

The student visualizes and illustrates ways in which shapes can be combined, subdivided, and changed. (MA.C.2.3, MA.C. 2. 4)

GOALS AND OBJECTIVES for

Florida's Frameworks for K-12 Gifted Learners

These new frameworks can used to enhance the level of rigor and intellectual challenge in classrooms of gifted learners.

1. By graduation, the student identified as gifted will be able to critically examine the complexity of knowledge: the location, definition, and organization of a variety of fields of knowledge.

   a) Locate, define, and organize a field of study as it relates to the broad spectrum of knowledge.
   b) Identify and illustrate basic principles and the foundational concepts that are central to understanding the essence of a field of study.
   b) Identify and apply investigative methodologies that are followed in a selected field of knowledge.

Course Outline / Overview

The National Council of Teachers of Mathematics (NCTM) has identified topology as one of the topics that should be better represented in the K-12 mathematics curriculum. In Canada, six to eight weeks in the middle grades is devoted to the introduction of the study of topological concepts. As an under-represented topic, the state objectives do not explicitly mention topology. However, it is a deductive system with axioms, theorems, and definitions that accompany its shapes. It most often appears in geometry materials and is usually relegated to the ancillary materials or the occasional sidebar reference. The Los Alamos National Laboratory has created an outstanding website with lessons for grades 5 and up to introduce students to these deep ideas which reflect scientific thinking.

The activities also help students increase their visualization and analysis of objects. Simple items such as knots and strips of paper take on new meanings and relationships. A three dimensional cube is in the same topological family as a paper triangle. Mathematics is the way we classify and organize out world.

The goals of this unit are to provide introductory activities for some of the more recognizable topological topics. A compilation of topics should make some of the applications of topology more readily apparent. The Mobius strip, for example, is used in some printer ribbons and mechanical belts to make the items last twice as long

Since I began working with these items in my classes in 1997, computers have made the study of topology more accessible. Computers can simulate shapes that are difficult to create. The online resources section provides an introduction to many topological concepts. Jeff Weeks has developed a packet of materials to accompany his Shape of Space video and activities. His ten online games help students to visualize topological concepts through the use of familiar games.
Other sites such as the Math Forum offer topological puzzles and problems.

Most lessons are accompanied by worksheets. The lesson plans contain answers to most of the worksheet problems as well as discussion notes and lesson extensions.

The topological activities in this packet were chosen to encourage investigation of patterns, critical thinking, and introduce a different perspective on the shapes around us. It introduces students to deep ideas in mathematics using simple objects. Each of the sections can stand alone or be joined into a longer unit. I use the items one or two at a time throughout the year. The Mobius strip investigation show students that objects are not always as they seem. The Knot investigation can be tied to left and right handed isotopes in chemistry. These are mirror image molecules that cannot be changed to the other form much like the two trefoil knots.
The networks have implications for the finding the shortest route, etc.

This packet is an introduction to various topological concepts. It may seem like a collection novelty lessons, but it contains deep mathematics.

Lesson Plans for Topology Classifications

Objective: Students will be able to use a classification system to separate objects according to their topological genus.

Materials: Topology Classifications worksheet OR an overhead acetate with the lesson.
Pencil

Online Resource: Encyclopedia Britannica animation of a doughnut morphing into a coffee cup
http://www.britannica.com/eb/article-258072/topology

Activities: Discuss topology as a form of geometry that allows items to stretch and twist. Objects that can be stretched into each other's form are called 'topologically equivalent' The tear allows the items to be twisted inside out. More tears allow more twists. The surface portions can be stretched or compacted infinitely.

If a computer is available, show the animation from the Encyclopedia Britannica. A coffee cup (genus 1 for one hole or tear) morphs into a doughnut (also genus 1).

Worksheet suggestions:

After the students have divided the numerals into three groups and written why they divided the numerals as they did. (Answer: According to the number of holes)

Group A: 1, 2, 3, 5, 7 *Group B*: 0, 4, 6,9 *Group C:* 8

Ask the students if they could write any numerals in a way that would change their group (genus.) The most likely choices are to place a loop on the bottom of the 2 and/or write the four with an open top.

In the second activity, pairs A- E are topologically equivalent. If you made one item from clay, you could morph it into the partner item without tearing additional holes.

Caution: The topological equivalence of pair F depends upon the type of ice cream cone. If a flat bottom cone is used, then the two items are NOT topologically equivalent. If a true cone shape (the kind that always leaks melted ice cream) is used, then the pair are topologically equivalent.

Extension: a) Have students find pictures or objects that belong to different topological genuses. A sheet of Xerox paper is genus 0. A sheet of notebook paper is genus 3 for the three holes in it.
b)Students play one of the ten familiar games on Jeff Weeks'website which is linked to NCTM's Library of Virtual Manipulatives. Instead of board, the games take place on a torus, Klein bottle, etc.
http://www.geometrygames.org/

Topology Classifications

According to Webster's New Universal Unabridged Dictionary, topology is: " The study of those properties of geometric figures that remain unchanged even when under distortion, so long as no surfaces are torn." Informally, topology has often been called "rubber sheet" geometry. This lesson is an overview of several topics in topology.

Topologically, the following numerals can be divided into three groups.

0 1 2 3 4 5 6 7 8 9

List your groupings:

Group A ____________ Group B ______________ Group C______________

Why did you group the numerals as you did?

1) Can you list one everyday object that could be included in each of the three groups?
   If two items belong to the same group, they are said to be 'topologically equivalent' to the other items in the group.

2) Which of these pairs are topologically equivalent? Explain.

A) A sewing needle and a coffee cup

B) A drinking glass and pencil

C) A square and a triangle

D) Eyeglass frames and a pair of scissors

E) A sheet of paper and a cube

F) An ice cream cone and a soda straw

3) Can you name another pair of items which are NOT topologically equivalent?

Lesson Plans for Networks

Objective: The student will determine if a network can be traversed or not.

Materials: Networks worksheet
Pencil

Activity: Students work in cooperative groups to complete the worksheet with periodic whole class discussions.

Notes to the teacher: Topology appears in the lesson with the shrinking of the riverbanks and islands to points (also called as nodes or vertices.) The seven Konigsberg bridges become curves (edges) which connect the nodes. The original bridge diagram and the 'shrunken' form are *topologically equivalent*.

Traversable diagrams occur when no more than two odd nodes (vertices) are present. The Konigsberg Bridges have four odd nodes and therefore are not traversable. When the bridge is added from X to Y, the riverbanks change from odd nodes with three edges each to even nodes with four edges each. (Edges connect nodes.)

Odd nodes are always the starting or ending point of a traversable network. Since there can be at most two starting/ending points, a traversable network can have no more than two odd nodes. Networks with 0, 1, or 2 nodes are always traversable. If a node is even, you can pass into and out of it each time you approach it. If it is odd, you have at one visit where you can enter and not leave (ending point) or leave and not re-enter without passing back out (starting point.)

In the house diagram, each room serves as a node. There are rooms (nodes) with 4, 5, 5, 4, and 5 doors (edges) entering them. Since there are more than 2 odd nodes, you *cannot* pass through each door exactly once. Could a door be added to allow you to pass through each door once?

Answers to **Can these puzzles be traced?**

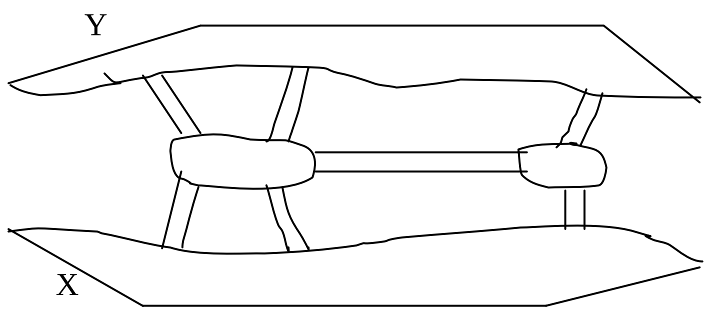
Yes (all even nodes), No (4 odd nodes), Yes (only two odd nodes), No (4 odd nodes)

No (4 odd nodes), Yes (2 odd nodes), Yes (all even nodes), No (4 odd nodes)

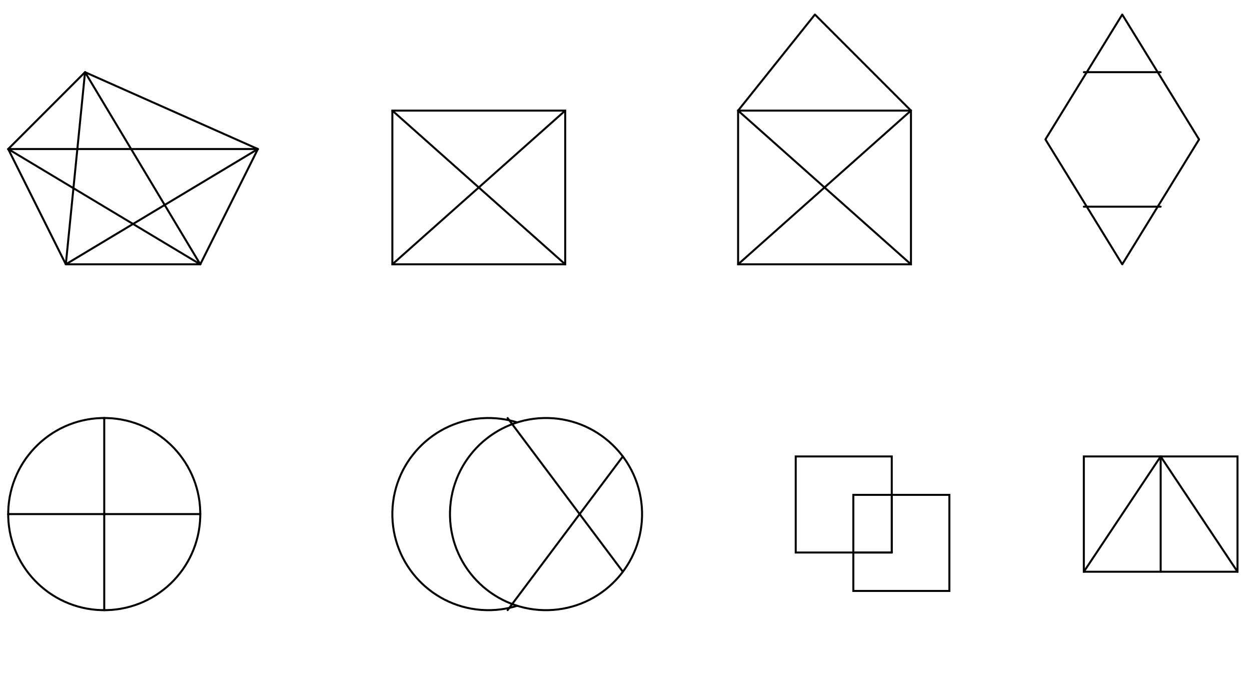
Extensions:
a) Add lines(edges) to make the networks on the bottom of side 1 traversable.
b) Determine if 3-D shapes such as cubes and other polyhedra are traversable.
c) Determine if traversable networks are circuits (all even) or paths (start/end at different nodes).
d) Research Euler circuits/paths versus Hamiltonian circuits/paths. Determine if the polyhedra have Euler and/or Hamiltonian circuits/paths.

Networks

Perhaps the best known of the network problems is The Konigsberg Bridge Problem which was first set forth by Leonhard Euler. In this problem, the citizens of Konigsberg take an evening stroll over the seven bridges which connect the two banks of the river and two islands in the middle of the river. The question posed is whether they can stroll over each bridge just once and arrive back at their starting place.

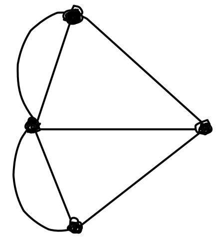


Before we can answer the bridge question, let us look at other network problems you have seen. Networks also appear as puzzles that ask, “Can you trace each line exactly once without lifting your pencil?” Try these. Which can be traced? Which cannot?



How can you tell if a network can be traced (is *traversable*) without actually tracing it?

Let us look again at the Konigsberg Bridge problem. Suppose we shrink the two islands and the two banks of the river to look like dots. We shrink the bridges to single curves.

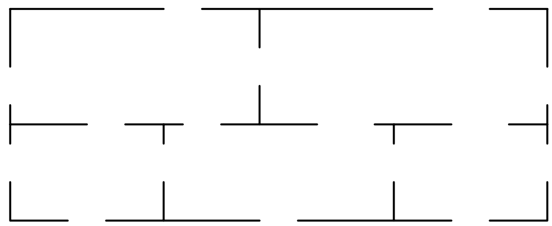


Topology allows us to stretch, twist, and bend items. We can distort distances. The Konigsberg Bridge is now a network. Can you trace a path over the bridges and solve the problem? Why?

Recently, another bridge has been added from point X to point Y in the original bridge diagram. Can the new Konigsberg bridges be traversed? Why or why not?

This is a more challenging and less obvious network problem.

Below is the floor plan of a house. Can you walk through each door exactly once?



Hint: This time let the rooms be points. Let the doorways be the lines that connect the points.

Is the house traversable? Why or why not?

Can you create a network with six nodes that is not traversable?

Lesson Plans for Mobius Strip Twists

Objective: To discover patterns in modifications of Mobius strips and related paper loops.
To use inductive reasoning to draw conclusions.

Materials: Mobius Strip Twist handout (follows)
Tape, gluesticks, or glue.
Strips of paper 2 inches by 18 inches - adding machine tape works well.
Scissors
Pencil, crayon, or marker

Procedures: Demonstrate how to make a one twist Mobius strip. It is important to secure the strip across the entire length of the joining so that the two ends remain joined after cuttings.

(Note: An odd number of twists will generate Mobius strip properties. An even number of twists result in ordinary loop behavior.)

In the second investigation, a cut down the center requires results in a smaller loop than the cut 1/3 of the way from a side. The cut 14 of the way from a side will generate an even larger loop.

The investigations lend themselves to jigsaw activities. Each group could be assigned to investigate / report one of the rows on the chart.
Assist the class in summarizing the results of the two investigations and in discussing what was discovered. The worksheet has questions for starting the discussion.

Extension

a) Make a Mobius loop and a zero twist loop. Tape them together at right angles so that the two loops are perpendicular at the point of intersection. Cut through the centers of both loops and through the intesection. What is the result?
For more information: http://www.math.wichita.edu/history/activities/geometry-act.html

b) Place two strips of paper on top of each other. Make one twist. Tape the top of one end of the twist to the top of the other end. Similarly connect the bottom of each twist. Slide a pencil between the two layers and run it around the loop. How many pieces of paper do you have now?

c) Find pictures of a Klein bottle and of a torus on the internet. The Klein bottle has no inside or outside. The torus (doughnut) is the classic genus 1 shape. Imagine if the world were in the shape of torus. Would aircraft be able to lift off in the inner region (the 'hole')?

Mobius Strip Twists

This is probably the best known of the topological shapes. Take a strip of paper. Give it one twist. Tape or glue the ends together. How many sides does the shape have? Check your guess by placing your pencil on the center of a side. Draw a line down the center until you get back to the point where you started.

What happens? How many sides does your Mobius strip have? Why do you think so?

What happens when you cut along the line that you drew? Call this shape "Cut 2"

What happens if you cut down the center of Cut 2?

What would happen if you repeated the experiment with two twists in the strip of paper?

Complete the following table. Remember to include a zero twist loop in your chart.

Number of Twists	Number of Sides	Pieces after 1 cut	Pieces after two cuts
0			
1			
2			
3			

What pattern do you see in the number of twists and the number of pieces?

What do you predict will occur when there are 4 twists in the strip?

Make a 4 twist loop and check your prediction. Was it a correct prediction?

Here's another Mobius strip exploration:

In the last investigation, we cut down the middle or halfway from the edge of the Mobius strip.

This time, draw a line one-third of the way from the edge of the one twist Mobius strip.
What happened?

Cut along that line. Did anything unexpected happen?

Draw another line 1/3 of the distance from the edge.

Cut along that line. What happened?

Complete the following chart. Each experiment should be with a one twist Mobius strip.

Distance from the edge	Number of loops Traveled	Number of loops after Cut 2
One-half of the distance		
One-third of the distance		
One-fourth of the distance		

Summarize what you discovered in this investigation:

Lesson Plans for Color My World

Objective: To introduce the Four Color Problem of maps

Materials: Color My World worksheet
Crayons or colored pencils
Scissors
Tape

Activities: Students complete the worksheet. Each diagram can be colored with 2-4 colors. A two-dimensional map or diagram can be colored with no more than 4 colors according to the Four Color problem. (An inspection of any atlas should support this.)

Students may need some assistance in understanding the rolling of tubes. Not all the diagrams can be rolled into tubes. Among those tubes that can be rolled, some tubes can be rolled north-south, some east-west, and some can be rolled in two different manners.

When the tubes are rolled, the original edges must still be considered to exist. For example, if two red areas come into contact when the tube is rolled, then the original two dimensional coloring pattern must be altered for the three dimensional tube. Note that the change to 3-D has moved the discussion out of the Four Color Problem's limitation on four colors. Some of the tubes might require more than four colors.

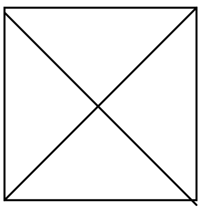
Extensions: a) Have students color maps of Africa, Asia, etc.
b) Have students continue the investigation with items from MegaMath

Color My World

Map Coloring

The Four Color Problem has shown that any two dimensional map can be colored using no more than four colors. However, a method has not been found to determine how many colors are needed to color a particular map. Look at an atlas. Can you find any maps with more than four colors? Try your hand at coloring each of these maps.

What is the least number of colors that are needed? Colors can touch at a point, but not along an edge or line.



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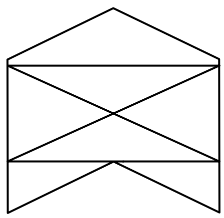

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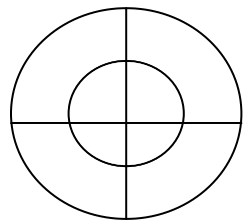

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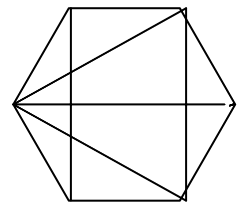

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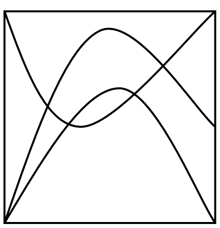
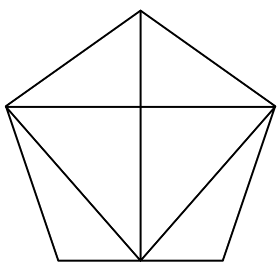


E



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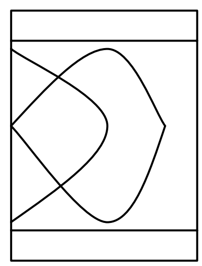
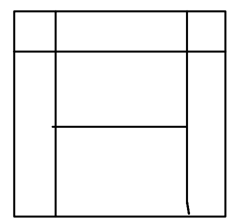
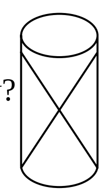


G



H

Suppose you cut out Figure A and rolled it into a tube.
If all the edges remain, how many colors will you need now?



Would the numbers of colors change if you rolled Figure H into a tube?
Hint: How many ways can you roll Figure H into a tube?

Are there any other figures that can be rolled into a tube without changing the colors?

Lesson Plans for Why Knot?

Objective: To identify the properties of knots as an introduction to knot theory, an emergent mathematical field of study
To link the properties of specific knots to science concepts.
To strengthen critical thinking
To increase visual perception and tactile skills

Materials: Why Knot? Worksheet plus the knots collection from MegaMath or other sources
http://www.c3.lanl.gov/mega-math/workbk/knot/knot8.gif
Crayons or colored pencils
24 inch lengths of cords suitable for tying /untying knots.
Optional: A book of different knots

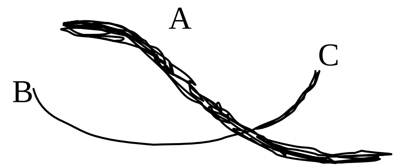
Activities: a) Have students color the knots according to the directions on the sheet.
b) Note the right and left trefoil knots on the MegaMath page.
These knots are mirror images of each other and cannot be pulled into the other's shape. A similar concept arises in chemistry where isotopes may be mirror images.
c) Have students create different knots using thin silken cords from a book of knots.

Extensions: See the MegaMath site
One extension has the students make bubble wands with one of the knots as the bubblemaking head. The knot is made from lightweight wire.

Why Knot?

Knot theory has many parts. The part we will explore in this lesson is similar to the map coloring problem. The coloring of a knot concerns the crossings in the knot. In each crossing there are three things to color:

A) The strand that crosses over
B) One side of the strand that goes under
C) The other side of the strand that goes under.



The coloring rules are:

a) You may color the strands so that they are each different colors at the crossing
   The knot must have this happen at each crossing.

or b) You may color the strands so they are all the same color at the crossing.

Color the strands as your teacher directs you.

NOTE: Copy knot diagrams onto this space . Use ones similar to those found at
http://www.c3.lanl.gov/mega-math/workbk/knot/knot8.gif
or create your own knot diagrams.

Can you make these knots using rope?

The Shape of Space Activities

These activities have been created by Jeff Weeks and presented over the past decade. The activities incorporate many topological concepts.

From the Torus Games website with free downloads: “Eight familiar games introduce children ages 10 and up to the mind-stretching possibility of a “multiconnected universe”. Games include: tic-tac-toe, mazes, crossword puzzles, word search puzzles, jigsaw puzzles, chess, pool and apples”
http://www.geometrygames.org/

The students and adults play familiar games using topological shapes. The games stimulate thinking in topological rather than in two or three-dimensions.

There is also a video and a workbook with ten lessons that go a little deeper into the topics I have introduced in this packet. I have not seen the book.

See the Books section at the end of the packet.

MegaMath at Los Alamos Laboratory

www.c3.lanl.gov/mega-math/welcome.html

“ Mathematics is a live science with new discoveries being made every day. The frontier of mathematics is an exciting place, where mathematicians experiment and play with creative and imaginative ideas. …The MegaMath project is intended to bring unusual and important mathematical ideas to **elementary school classrooms** so that young people and their teachers can think about them together.” From the MegaMath site.

This is an exemplary site created as an K-12 outreach by the Los Alamos National Laboratory. There are complete lesson plans for teachers with additional links to National Council of Teachers of Mathematics (NCTM) materials.

<u>The lesson plan titles are</u>:

The Most Colorful Math of All
Games on Graphs
Untangling the Mathematics of Knots
Algorithms and Ice Cream for All
Machines that Eat Your Words
Welcome to the Hotel Infinity
A Usual Day at Unusual School

Classic Topological Games and Puzzles

There are many topological puzzles. Most of the metal and string puzzles found in puzzle shops are topologically based. The pencil/ buttonhole problem is one of the most common. (See the online resources section.)

In 1967, a new pencil and paper game, the Game of Sprouts, was created by two mathematicians while studying at Cambridge. The games consists of making nodes and arcs according to short set of rules. It is the topological equivalent of the dots and squares game played by most schoolchildren.

Most topological puzzles involve using distortions of nodes and edges as we saw in the "Can you pass through every door once ?" network puzzle. The rooms had to be shrunken to points and the doorways became edges or lines connecting them. The pencil/buttonhole problem does this as do most string/metal puzzles.

Some of the loops in the children's game of Cat's Cradle are topological knots

An old parlor trick that I first saw in the original Matchmaker movie with Shirley Booth is the 'string handcuff puzzle." It requires two persons and two four foot pieces of string/twine. One person ties a loop in end of the string and slips one loop over each hand. The second person does the same **except** she/he first loops the string through the drooping string of the first person. The drooping parts of each 'handcuff' loop through each other once. The object is to separate the strings without removing the loops from any wrist. (Answer below.)

Solution to the string handcuffs above: Take the crossed string, make a u-bend, and slip it up through the inner wrist area of one person and then open it over the hand without twisting the string. Pull gently and the strings should separate.

Online Resources

<u>TOPOLOGY</u>

R-U-B-B-E-R Geometry by Jill Britton
Ms. Britton has myriad topological activities
She also writes materials for Creative Publications/Dale Seymour.
http://britton.disted.camosun.bc.ca/jbrubbergeom.htm

The Seven Bridges of Konigsberg Problem
http://mathforum.org/isaac/problems/bridges1.html

Introductory Activities with Networks
http://www1.conyers.stockton.sch.uk/sparkingthegap/maths/euler/nodes.asp

Website linking origami and networks
http://www.paperfolding.com/math/

Mobius explanation and history
http://www.math.wichita.edu/history/topics/geometry.html#mobius

Encyclopedia Britannica-animation of a doughnut morphing into a coffee cup
http://www.britannica.com/eb/article-258072/topology

MegaMath Topology-an Outreach of the Los Alamos Laboratory
http://www.c3.lanl.gov/mega-math/gloss/topo/topo.html

Description of topology and its uses/study-intro to college course
http://www.stetson.edu/~mhale/topology/index.htm

Website of a classroom that used The Shape of Space
Contains links to many more sources.
http://www.mccallie.org/myates/the_shape_of_space.

Interactive animated Moebius strip
http://www.ch.ic.ac.uk/rzepa/talks/acs06/mobius1.html

The Math Forum's Topology Resource list
http://mathforum.org/library/topics/topology/

<u>KNOT THEORY</u>

The Math Forum Knot Theory resources
http://mathforum.org/library/topics/knot_theory/

Megamath Introduction to Knot Theory for students by the Los Alamos Laboratory
http://www.c3.lanl.gov/mega-math/workbk/knot/knot.html

TOPOLOGICAL PUZZLES

The Pencil and Buttonhole Topological Puzzle
http://www.aimsedu.org/Puzzle/pencil/pencil.html

The Game of Sprouts - a illustrated explanation of how to play.
http://www.madras.fife.sch.uk/maths/games/sprouts.html

The Torquato Puzzle of Fibonacci
http://www.archimedes-lab.org/workshoptorquato.html

Hexaflexagons –a topological variation
http://www.murderousmaths.co.uk/games/flex/flexmake.htm

TOPOLOGICAL INVESTIGATIONS

Eight familiar games to introduce children age 10 and up to the concept of a finite yet unbounded universe from Jeff Weeks' website. The games include Torus and Klein Bottle activities.
http://www.geometrygames.org/

Books

The Shape of Space Activity Book by Jeff Weeks is a series of 10 lessons that support the Shape Of Space video. Key Curriculum Press http://www.keypress.com/x5876.xml

Exploring the Shape of Space Book/CD/Video, U.S. Format (NTSC)
ISBN: 978-1-55953-469-7 | Bundle: Softcover/CD/VHS $50.95

Activity Book alone with digitized version of the Shape of Space video
ISBN: 978-1-55953-467-3 | 133 pp. | Bundle: Softcover/CD $33.95

The Mobius Strip Dr August Mobius' Marvelous Band in Mathematics, Games, Literature, Art, Technology, and Cosmology; Pickover, Clifford A.; Thunder Mouth's Press, New York, 2006
ISBN 978-0-7394-754-3 Softcover $15.95
As the title states- Mobius in the world around us. It discusses the Mobius strip in art, literature, and industry along with other topological shapes. It is not for students.
More info: http://sprott.physics.wisc.edu/Pickover/mobius-book.html

Bibliography

Internet Sources

http://turnbull.mcs.st-and.ac.uk/history/Indexes/Geometry_Topology.html
Topology site at St.Andrews College in Scotland

www.c3.lanl.gov/mega-math/welcome.html
Fantastic topology site by Los Alamos National Laboratory with complete lesson plans for teachers including NCTM links .

Textbooks:

UCSMP Geometry; University of Chicago; Scott Foresman Addison Wesley, 1997.

Geometry; Jurgensen et al, Houghton Mifflin, 1986.

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This booklet is out of print but appears on online used book sources. It remains an excellent source of topological activities